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The stability of short waves on a straight vortex filament in a weak externally imposed strain field

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The stability of short-wave displacement perturbations on a vortex filament of constant vorticity in a weak externally imposed strain field is considered. The circular cross-section of the vortex filament in this straining flow field becomes elliptical. It is found that instability of short waves on this strained vortex can occur only for wavelengths and frequencies at the intersection points of the dispersion curves for an isolated vortex. Numerical results show that the vortex is stable at some of these points and unstable at others. The vortex is unstable at wavelengths for which $\omega = 0$, thus giving some support to the instability mechanism for the vortex ring proposed recently by Widnall, Bliss & Tsai (1974). The growth rate is calculated by linear stability theory. The previous work of Crow (1970) and Moore & Saffman (1971) dealing with long-wave instabilities is discussed as is the very recent work of Moore & Saffman (1975).

1. Introduction

In recent years, there has been a renewed interest in the behaviour of flows with concentrated vorticity, in particular in the stability of various configurations of vortex filaments: vortex pairs, vortex rings and isolated vortex filaments. The recent review by Widnall (1975) discusses a number of these problems. The present work is motivated by the work of Widnall *et al.* (1974), who proposed a simple model for the instability of a vortex ring suggesting that instability would occur on the vortex ring at wavelengths for which the waves on a corresponding straight filament would not rotate ($\omega = 0$). The present paper analyses the more tractable but related problem of the stability of short waves on a vortex filament in a flow field with strain. The physical model presented in Widnall *et al.* (1974) implies that the straight filament would also be unstable under these conditions.

Thompson (1880) investigated the vibration of an isolated columnar vortex with solid-body rotation surrounded by an irrotational revolving flow and showed that it is stable to all infinitesimal disturbances, which merely rotate around the filament. Crow (1970) analysed the stability of a pair of trailing vortices to long-wave displacement perturbations by considering the mutual interaction of a pair of sinusoidally perturbed vortex lines: the Biot–Savart law was used to calculate the velocity induced at each vortex by the presence and deformation of the other vortex and the 'cut-off' method was used to calculate the self-induced motion of the perturbed vortex itself. In a co-ordinate system fixed to one of the vortices in an undisturbed descending vortex pair, a strain field is produced by the other vortex. Instability occurs for long waves when the self-induced rotation rate becomes small and the displacement perturbations diverge in the strain field. Widnall, Bliss & Zalay (1971) first determined the effect of vortex-core structure on the self-induced rotation of a thin curved vortex filament; the general solution found by them was then applied to study the stability of a vortex pair. Both of these analyses showed that there are two groups of unstable waves: long waves, for which the maximum non-dimensional amplification rate [$\bar{\alpha} = \alpha/(\Gamma/2\pi b^2)$] is 0·8, and short waves, for which $\bar{\alpha}$ is 1. The short-wave instability occurs at a wavelength for which the long-wave self-induction theory incorrectly predicts $\omega = 0$.

Recently, Widnall *et al.* (1974) have re-examined the existence of short waves on straight filaments in connexion with study of a short-wave instability of the vortex ring and concluded that this prediction of short-wave instability is spurious (unless a strong axial velocity exists in the core) but that short-wave instabilities having a more complex radial structure can exist. (See also Widnall 1975.) A more complete analysis of the dispersion relation for waves on straight filaments without a long-wave assumption showed that the higher-order radial modes of bending the filament have the property that for some wavenumber the self-induced rotation rate goes to zero. They postulated that it is these modes having ω close to zero that diverge in the straining flow field. Results of a calculation based upon this simple model of second-radial-mode instability showed good agreement with the experimentally obtained wavenumbers for unstable vortex rings.

However, a complete analysis of this instability for either the ring or the line filament requires that small perturbations of the *actual* steady flow be considered. For the straight filament considered here, the cross-section of the region of constant vorticity in a straining flow becomes an ellipse; the velocity field is no longer purely tangential. Moore & Saffman (1971) examined the stability of this vortex in two special cases: two-dimensional disturbances and long-wave disturbances. They showed that this flow is always unstable to displacement perturbations which are independent of the axial co-ordinate; the growth rate is equal to the rate of strain. For long-wave disturbances, three-dimensional effects reduce the growth rate of the instability, confirming the long-wave analysis of Crow (1970). During revision of this paper, a paper with the title "The instability of a straight vortex filament in a strain field", by Moore & Saffman (1975), was submitted for publication. In this, they establish that a vortex would be unstable in the presence of strain if it could support non-rotating waves, but no specific solutions or numerical results are presented.

In order to complete the stability theory for the vortex line and to give support to the instability mechanism for the vortex ring proposed by Widnall *et al.* (1974), we have investigated the stability of short bending waves on a vortex filament of constant vorticity with finite core size in a weak straining flow field defined by the small parameter $\epsilon = 4e/\Omega$, where e is the rate of strain and Ω is the vorticity inside the core.

2. Formulation of the problem

We formulate the problem by expanding both the steady and unsteady flow solution in the small strain parameter ϵ (§2), then introduce three-dimensional disturbances along the axial direction, derive a solvability condition required for instability and present some numerical results (§3).

The asymptotic steady-flow solution for the structure of a line vortex with a circular cross-section in a weak straining flow field can be obtained by expanding the solution as a power series in ϵ and imposing kinematic and dynamic matching conditions expanded in powers of ϵ at the edge of the vortex core or by directly expanding the exact solution obtained by Moore & Saffman (1971). The result is

$$\left.\begin{array}{l} \left.\begin{array}{l} U(r,\theta) = -\epsilon r \sin 2\theta + O(\epsilon^2), \\ V(r,\theta) = r - \epsilon r \cos 2\theta + O(\epsilon^2), \\ P(r,\theta) = \tfrac{1}{2}r^2 - 1 + O(\epsilon^2), \end{array}\right\} \quad 0 \leqslant r \leqslant R(\theta,\epsilon), \\ \Phi(r,\theta) = \theta + \tfrac{1}{4}\epsilon(r^{-2} - r^2)\sin 2\theta + O(\epsilon^2), \quad R(\theta,\epsilon) \leqslant r < \infty, \end{array}\right\} \tag{2.1}$$

where $R(\theta,\epsilon)$ denotes the boundary of the vortex core:

$$R(\theta,\epsilon) = 1 + \tfrac{1}{2}\epsilon \cos 2\theta + O(\epsilon^2).$$

U, V, P and Φ are the radial velocity, tangential velocity and pressure inside the vortex core and the velocity potential outside the vortex core, normalized as follows: the velocities by $\frac{1}{2}\Omega a$, pressure by $\frac{1}{4}\rho\Omega^2a^2$ and velocity potential by $\frac{1}{2}\Omega a^2$; a is the radius of the vortex core.

We subject this basic flow field to small perturbations (denoted by tildes) and linearize the equations of motion to obtain governing equations for the disturbance quantities: for the flow inside the vortex,

$$\left.\begin{array}{c} \dfrac{\partial \tilde{u}}{\partial t} + U\dfrac{\partial \tilde{u}}{\partial r} + \dfrac{\partial U}{\partial r}\tilde{u} + \dfrac{V}{r}\dfrac{\partial \tilde{u}}{\partial \theta} + \dfrac{1}{r}\dfrac{\partial U}{\partial \theta}\tilde{v} - \dfrac{2V\tilde{v}}{r} = \dfrac{-1}{\rho}\dfrac{\partial \tilde{P}}{\partial r}, \\ \dfrac{\partial \tilde{v}}{\partial t} + U\dfrac{\partial \tilde{v}}{\partial r} + \dfrac{\partial V}{\partial r}\tilde{u} + \dfrac{V}{r}\dfrac{\partial \tilde{v}}{\partial \theta} + \dfrac{1}{r}\dfrac{\partial V}{\partial \theta}\tilde{v} + \dfrac{U\tilde{v} + V\tilde{u}}{r} = \dfrac{-1}{\rho}\dfrac{1}{r}\dfrac{\partial \tilde{P}}{\partial \theta}, \\ \dfrac{\partial \tilde{w}}{\partial t} + U\dfrac{\partial \tilde{w}}{\partial r} + \dfrac{V}{r}\dfrac{\partial \tilde{w}}{\partial \theta} = -\dfrac{1}{\rho}\dfrac{\partial \tilde{P}}{\partial z}, \\ \dfrac{\partial \tilde{u}}{\partial r} + \dfrac{\tilde{u}}{r} + \dfrac{1}{r}\dfrac{\partial \tilde{v}}{\partial \theta} + \dfrac{\partial \tilde{w}}{\partial z} = 0; \end{array}\right\} \tag{2.2a}$$

for the flow outside the vortex,

$$\nabla^2\tilde{\phi} = 0. \tag{2.2b}$$

The boundary conditions are as follows:

(i) As $r \to \infty$ the disturbances decay.

(ii) At $r = 0$ the solution is non-singular.

(iii) The kinematic boundary condition at $r = R(\theta,\epsilon) + \delta\tilde{F}(t,\theta,z)$ is

$$D(r - R - \delta\tilde{F})/Dt = 0,$$

where $\delta\tilde{F}$ is the displacement of the edge of the core.

(iv) The dynamic boundary condition is that the pressure is continuous at

$$r = R(\theta,\epsilon)+\delta\tilde{F}(t,\theta,z).$$

After linearization, the kinematic boundary condition becomes

$$\frac{\partial \tilde{F}}{\partial t}+\frac{1}{R^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial \tilde{F}}{\partial \theta}+\tilde{F}\left(\frac{1}{R^2}\frac{dR}{d\theta}\frac{\partial^2 \Phi}{\partial r\,\partial \theta}-\frac{2}{R^3}\frac{dR}{d\theta}\frac{\partial \Phi}{\partial \theta}-\frac{\partial^2 \Phi}{\partial r^2}\right)$$
$$+\frac{1}{R^2}\frac{dR}{d\theta}\frac{\partial \tilde{\phi}}{\partial \theta}-\frac{\partial \tilde{\phi}}{\partial r}=0 \quad \text{at} \quad r=R, \qquad (2.3a)$$

$$\frac{\partial \tilde{F}}{\partial t}+\frac{V}{R}\frac{\partial \tilde{F}}{\partial \theta}+\tilde{F}\left(\frac{1}{R}\frac{dR}{d\theta}\frac{\partial V}{\partial r}-\frac{1}{R^2}\frac{dR}{d\theta}V-\frac{\partial U}{\partial r}\right)+\frac{1}{R}\frac{dR}{d\theta}\tilde{v}-\tilde{u}=0 \quad \text{at} \quad r=R \qquad (2.3b)$$

and the dynamic boundary condition becomes

$$\tilde{P}=-\tilde{F}\left[\frac{\partial P}{\partial r}+\frac{\partial \Phi}{\partial r}\frac{\partial^2 \Phi}{\partial r^2}+\frac{1}{R^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial^2 \Phi}{\partial r\,\partial \theta}-\frac{1}{R^3}\left(\frac{\partial \Phi}{\partial \theta}\right)^2\right]$$
$$-\frac{\partial \tilde{\phi}}{\partial t}-\frac{\partial \Phi}{\partial r}\frac{\partial \tilde{\phi}}{\partial r}-\frac{1}{R^2}\frac{\partial \Phi}{\partial \theta}\frac{\partial \tilde{\phi}}{\partial \theta} \quad \text{at} \quad r=R. \qquad (2.4)$$

U, V, P, Φ and R [from (2.1)] are then substituted into the above equations and perturbation quantities are expanded in ϵ, the strain parameter, in the form

$$(\tilde{u};\tilde{v};\tilde{w};\tilde{P}/\rho;\tilde{\phi};\tilde{F}) = (\tilde{u}_0+\epsilon\tilde{u}_1+\ldots;\tilde{v}_0+\epsilon\tilde{v}_1+\ldots;\tilde{w}_0+\epsilon\tilde{w}_1+\ldots;$$
$$\tilde{\pi}_0+\epsilon\tilde{\pi}_1+\ldots;\tilde{\phi}_0+\epsilon\tilde{\phi}_1+\ldots;\tilde{F}_0+\epsilon\tilde{F}_1+\ldots)\,e^{\omega t+ikz},$$

where k is the wavenumber along the z axis.

The eigenvalue ω is expanded as

$$\omega = \omega_0+\epsilon\omega_1+\epsilon^2\omega_2+\ldots.$$

If ω_1 is real and positive, the flow is unstable. To investigate the stability in the neighbourhood of $\omega=\omega_0$, the wavenumber k is also expanded:

$$k = k_i+\epsilon\tilde{k},$$

where k_i is a solution to the dispersion relation for $\omega=\omega_0$.

3. Stability analysis

We consider disturbances along the axial direction of wavelength comparable to the core size of the vortex, i.e. $ka = O(1)$: the disturbed flow is totally three-dimensional. The equations (2.2) governing the flow separate to different orders in the strain parameter ϵ as follows.

To zeroth order, we have

$$\left.\begin{aligned}\omega_0\tilde{u}_0+\frac{\partial\tilde{u}_0}{\partial\theta}-2\tilde{v}_0+\frac{\partial\tilde{\pi}_0}{\partial r}&=0,\\ \omega_0\tilde{v}_0+2\tilde{u}_0+\frac{\partial\tilde{v}_0}{\partial\theta}+\frac{1}{r}\frac{\partial\tilde{\pi}_0}{\partial\theta}&=0,\\ \omega_0\tilde{w}_0+\partial\tilde{w}_0/\partial\theta+ik_i\tilde{\pi}_0&=0,\\ \frac{\partial\tilde{u}_0}{\partial r}+\frac{\tilde{u}_0}{r}+\frac{1}{r}\frac{\partial\tilde{v}_0}{\partial\theta}+ik_i\tilde{w}_0&=0,\end{aligned}\right\}\quad\text{for}\quad r<1, \tag{3.1}$$

$$\frac{\partial^2\tilde{\phi}_0}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{\phi}_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\tilde{\phi}_0}{\partial\theta^2}-k_i^2\tilde{\phi}_0=0\quad\text{for}\quad r>1 \tag{3.2}$$

and the boundary conditions at $r=1$ are

$$\partial\tilde{\phi}_0/\partial r-u_0=0, \tag{3.3a}$$

$$\tilde{\pi}_0+\omega_0\tilde{\phi}_0+\partial\tilde{\phi}_0/\partial\theta=0. \tag{3.3b}$$

To first order, we have

$$\left.\begin{aligned}\omega_0\tilde{u}_1+\frac{\partial\tilde{u}_1}{\partial\theta}-2\tilde{v}_1+\frac{\partial\tilde{\pi}_1}{\partial r}&=-\omega_1\tilde{u}_0\\ &\quad+\left(r\frac{\partial\tilde{u}_0}{\partial r}+\tilde{u}_0\right)\sin 2\theta+\frac{\partial\tilde{u}_0}{\partial\theta}\cos 2\theta,\\ \omega_0\tilde{v}_1+2\tilde{u}_1+\frac{\partial\tilde{v}_1}{\partial\theta}+\frac{1}{r}\frac{\partial\tilde{\pi}_1}{\partial\theta}&=-\omega_1\tilde{v}_0\\ &\quad+\left(r\frac{\partial\tilde{v}_0}{\partial r}-\tilde{v}_0\right)\sin 2\theta+\left(2\tilde{u}_0+\frac{\partial\tilde{v}_0}{\partial\theta}\right)\cos 2\theta,\\ \omega_0\tilde{w}_1+\frac{\partial\tilde{w}_1}{\partial\theta}+ik_i\tilde{\pi}_1&=-\omega_1\tilde{w}_0+r\frac{\partial\tilde{w}_0}{\partial r}\sin 2\theta\\ &\quad+\frac{\partial\tilde{w}_0}{\partial\theta}\cos 2\theta-i\tilde{k}\tilde{\pi}_0,\\ \frac{\partial\tilde{u}_1}{\partial r}+\frac{\tilde{u}_1}{r}+\frac{1}{r}\frac{\partial\tilde{v}_1}{\partial\theta}+ik_i\tilde{w}_1&=-i\tilde{k}w_0,\end{aligned}\right\}\quad\text{for}\quad r<1, \tag{3.4}$$

$$\frac{\partial^2\tilde{\phi}_1}{\partial r^2}+\frac{1}{r}\frac{\partial\tilde{\phi}_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\tilde{\phi}_1}{\partial\theta^2}-k_i^2\tilde{\phi}_1=2k_i\tilde{k}\tilde{\phi}_0\quad\text{for}\quad r>1 \tag{3.5}$$

and the boundary conditions at $r=1$ are

$$\frac{\partial\tilde{\phi}_1}{\partial r}-\tilde{u}_1=\left(-\frac{\partial\tilde{\phi}_0}{\partial\theta}+\tilde{v}_0\right)\sin 2\theta+\tfrac{1}{2}\left(-\frac{\partial^2\tilde{\phi}_0}{\partial r^2}+\frac{\partial\tilde{u}_0}{\partial r}\right)\cos 2\theta, \tag{3.6a}$$

$$\tilde{\pi}_1+\omega_0\tilde{\phi}_1+\frac{\partial\tilde{\phi}_1}{\partial\theta}=-\omega_1\tilde{\phi}_0+\frac{\partial\tilde{\phi}_0}{\partial r}\sin 2\theta+\tfrac{1}{2}\left(2\frac{\partial\tilde{\phi}_0}{\partial\theta}-\omega_0\frac{\partial\tilde{\phi}_0}{\partial r}-\frac{\partial^2\tilde{\phi}_0}{\partial\theta\,\partial r}-\frac{\partial\tilde{\pi}_0}{\partial r}\right)\cos 2\theta. \tag{3.6b}$$

The zeroth-order problem is of course that for waves on a straight filament; solution of (3.1) and (3.2) with the application of the boundary conditions (3.3) will determine ω_0, the zeroth-order frequency. Note that the first-order problem (3.4)–(3.6) is a forced problem whose homogeneous solution is identical to the zeroth-order solution with the zeroth-order eigenvalue ω_0 already determined; the unknown ω_1 that appears in the forcing terms is then determined by solvability conditions governing the forced eigenvalue problem.

Zeroth-order disturbances are assumed in the form

$$\left.\begin{aligned} \tilde{u}_0 &= A_0(r)\,e^{i\theta} + \bar{A}_0(r)\,e^{-i\theta}, \quad \tilde{v}_0 = B_0(r)\,e^{i\theta} + \bar{B}_0(r)\,e^{-i\theta}, \\ \tilde{w}_0 &= C_0(r)\,e^{i\theta} + \bar{C}_0(r)\,e^{-i\theta}, \\ \tilde{\pi}_0 &= D_0(r)\,e^{i\theta} + \bar{D}_0(r)\,e^{-i\theta}, \quad \tilde{\phi}_0 = E_0(r)\,e^{i\theta} + \bar{E}_0(r)\,e^{-i\theta}. \end{aligned}\right\} \tag{3.7}$$

The solution to this problem leads to the well-known dispersion relation for waves on an isolated straight vortex filament (e.g. Moore & Saffman 1971). We outline the procedure in some detail to highlight the solution for a vortex filament in the presence of strain. Substituting the assumed disturbances (3.7) into the set of equations (3.1) and equating the coefficients of $e^{i\theta}$ and $e^{-i\theta}$, we obtain two sets of equations of motion. By manipulation, these can be reduced to two simultaneous ordinary differential equations governing only the pressure; these two equations can be written as

$$L_1 D_0 = 0, \quad L_2 \bar{D}_0 = 0, \tag{3.8}$$

where the operators are

$$L_1 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - \eta_1^2,$$

$$L_2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - \eta_2^2,$$

with

$$\eta_1^2 = -k_i^2(\omega_0 - i)(\omega_0 + 3i)/(\omega_0 + i)^2,$$

$$\eta_2^2 = -k_i^2(\omega_0 + i)(\omega_0 - 3i)/(\omega_0 - i)^2.$$

The solution to (3.8) for the pressure is then

$$D_0 = J_1(\eta_1 r)\,\beta_0, \quad \bar{D}_0 = J_1(\eta_2 r)\,\bar{\beta}_0, \tag{3.9}$$

where we have required that the solution is non-singular at $r = 0$; β_0 and $\bar{\beta}_0$ are arbitrary constants. The velocity components can be found from the pressure.

From (3.2) and (3.7) the velocity potential outside the vortex is given by

$$E_0 = K_1(k_i r)\,\alpha_0, \quad \bar{E}_0 = K_1(k_i r)\,\bar{\alpha}_0, \tag{3.10}$$

where K_1 is a modified Bessel function.

The homogeneous boundary conditions (3.3*a*) and (3.3*b*) at $r = 1$ become

$$\begin{bmatrix} k_i K_1'(k_i) & -\mathscr{A}_0 \\ (\omega_0 + i)\,K_1(k_i) & J_1(\eta_1) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} = 0, \tag{3.11a}$$

$$\begin{bmatrix} k_i K_1'(k_i) & -\bar{\mathscr{A}}_0 \\ (\omega_0 - i)\,K_1(k_i) & J_1(\eta_2) \end{bmatrix} \begin{bmatrix} \bar{\alpha}_0 \\ \bar{\beta}_0 \end{bmatrix} = 0, \tag{3.11b}$$

where

$$\mathscr{A}_0 = [-(\omega_0+i)\,\eta_1 J_0(\eta_1)+(\omega_0-i)\,J_1(\eta_1)]/[(\omega_0-i)\,(\omega_0+3i)],$$
$$\bar{\mathscr{A}}_0 = [-(\omega_0-i)\,\eta_2 J_0(\eta_2)+(\omega_0+i)\,J_1(\eta_2)]/[(\omega_0+i)\,(\omega_0-3i)].$$

The existence of a non-trivial solution (α_0, β_0) and/or $(\bar{\alpha}_0, \bar{\beta}_0)$ of (3.11*a*) and/or (3.11*b*) requires that the determinants of (3.11*a*) and/or (3.11*b*), denoted by $a(\omega_0, k_i)$ and $b(\omega_0, k_i)$, equal zero; ω_0 and k_i values satisfying these zeroth-order dispersion relations are shown in figure 1. From (3.11) and (3.12), we obtain

$$a(\omega_0, k_i) \equiv (\omega_0+i)\,K_1\mathscr{A}_0 + k_i K_1' J_1(\eta_1),$$
$$b(\omega_0, k_i) \equiv (\omega_0-i)\,K_1\bar{\mathscr{A}}_0 + k_i K_1' J_1(\eta_2).$$

Note that $a(\omega_0, k_i) = b(-\omega_0, k_i)$. Thus, from (3.11) and (3.12), the eigenmode is given by

$$\alpha_0 = \frac{\mathscr{A}_0}{k_i K_1}\beta_0 \quad \text{and/or} \quad \bar{\alpha}_0 = \frac{\bar{\mathscr{A}}_0}{k_i K_1}\bar{\beta}_0. \tag{3.12}$$

If (ω_0, k_i) are such that $a(\omega_0, k_i) = 0$ but $b(\omega_0, k_i) \neq 0$, then $\bar{\beta}_0 = 0$; if (ω_0, k_i) are such that $b(\omega_0, k_i) = 0$ but $a(\omega_0, k_i) \neq 0$, then $\beta_0 = 0$; if (ω_0, k_i) are such that $a(\omega_0, k_i) = 0$ and $b(\omega_0, k_i) = 0$ simultaneously, then β_0 and $\bar{\beta}_0$ are arbitrary: two eigenmodes of the same wavelength and equal and opposite frequencies can exist simultaneously. These (ω_0, k_i) points will be shown to have special significance for a vortex in the presence of strain.

To find the effect of strain on the dispersion equation, we proceed to the next order: to ω_1. Since the presence of strain changes the basic flow as given in (2.1), the disturbances to the next order are assumed to be of the form

$$\left.\begin{aligned}
\tilde{u}_1 &= A_{10}(r)+A_{11}(r)\,e^{i\theta}+\bar{A}_{11}(r)\,e^{-i\theta}+A_{12}(r)\,e^{2i\theta}+\bar{A}_{12}(r)\,e^{-2i\theta},\\
\tilde{v}_1 &= B_{10}(r)+B_{11}(r)\,e^{i\theta}+\bar{B}_{11}(r)\,e^{-i\theta}+B_{12}(r)\,e^{2i\theta}+\bar{B}_{12}(r)\,e^{-2i\theta},\\
\tilde{w}_1 &= C_{10}(r)+C_{11}(r)\,e^{i\theta}+\bar{C}_{11}(r)\,e^{-i\theta}+C_{12}(r)\,e^{2i\theta}+\bar{C}_{12}(r)\,e^{-2i\theta},\\
\tilde{\pi}_1 &= D_{10}(r)+D_{11}(r)\,e^{i\theta}+\bar{D}_{11}(r)\,e^{-i\theta}+D_{12}(r)\,e^{2i\theta}+\bar{D}_{12}(r)\,e^{-2i\theta},\\
\tilde{\phi}_1 &= E_{10}(r)+E_{11}(r)\,e^{i\theta}+\bar{E}_{11}(r)\,e^{-i\theta}+E_{12}(r)\,e^{2i\theta}+\bar{E}_{12}(r)\,e^{-2i\theta}.
\end{aligned}\right\} \tag{3.13}$$

By almost the same procedure as led to (3.8), we reduce (3.4) to two inhomogeneous simultaneous ordinary differential equations containing the pressure:

$$\begin{aligned}
L_1 D_{11} = {} & \left[\omega_1\left(\frac{-8}{(\omega_0+i)^3}k_i^2 J_1(\eta_1 r)\right)+\frac{2(\omega_0-i)\,(\omega_0+3i)}{(\omega_0+i)^2}J_1(\eta_1 r)\,k_i\tilde{k}\right]\beta_0\\
& +\left[\frac{-4\omega_0 i k_i^2}{(\omega_0-i)^2\,(\omega_0+i)^2}\eta_2 r J_0(\eta_2 r)+\frac{ik_i^2(\omega_0-3i)}{(\omega_0-i)^2}J_1(\eta_2 r)\right]\bar{\beta}_0,
\end{aligned} \tag{3.14a}$$

$$\begin{aligned}
L_2 \bar{D}_{11} = {} & \left[\omega_1\left(\frac{-8}{(\omega_0-i)^3}k_i^2 J_1(\eta_2 r)\right)+\frac{2(\omega_0+i)\,(\omega_0-3i)}{(\omega_0-i)^2}J_1(\eta_2 r)\,k_i\tilde{k}\right]\bar{\beta}_0\\
& +\left[\frac{4\omega_0 i k_i^2}{(\omega_0-i)^2\,(\omega_0+i)^2}\eta_1 r J_0(\eta_1 r)-\frac{ik_i^2(\omega_0+3i)}{(\omega_0+i)^2}J_1(\eta_1 r)\right]\beta_0.
\end{aligned} \tag{3.14b}$$

The inhomogeneous terms in (3.14) are found by substituting the zeroth-order solution into (3.4).

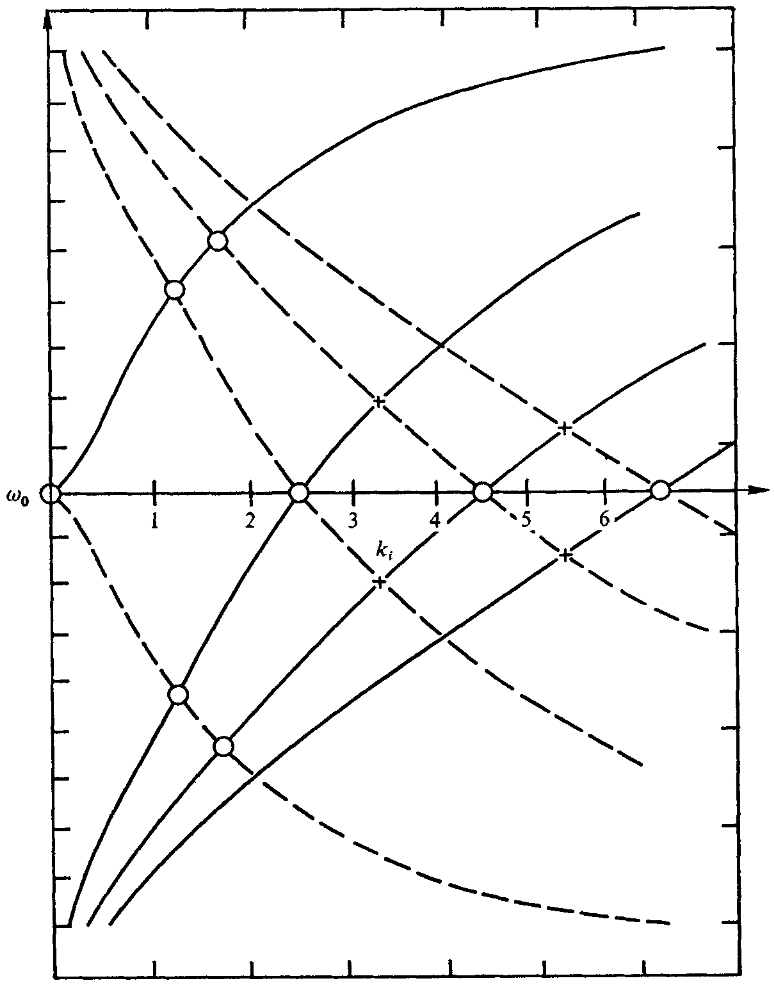

FIGURE 1. Curves of the dispersion relation for waves on a straight vortex filament. —, $a(\omega_0, k_i) = 0$; – – –, $b(\omega_0, k_i) = 0$. Crossing points indicate regions of possible instability to the presence of strain; ○, ω_1 positive and real, flow unstable; +, ω_1 imaginary, flow stable.

By standard methods, the solution to (3.14) for the first-order disturbance pressure is

$$D_{11} = J_1(\eta_1 r)\beta_1 + (\omega_1 H_1 + k_i \tilde{k} H_2)\beta_0 + \bar{H}_3\bar{\beta}_0, \tag{3.15a}$$

$$\bar{D}_{11} = J_1(\eta_2 r)\bar{\beta}_1 + (\omega_1 \bar{H}_1 + k_i \tilde{k} \bar{H}_2)\bar{\beta}_0 + H_3\beta_0. \tag{3.15b}$$

We have required the solution to be non-singular at $r = 0$ and β_1 and $\bar{\beta}_1$ are arbitrary constants. The terms introduced in (3.15) are defined in appendix A. The disturbance velocity field can be calculated from the solution for the disturbance pressure field.

Applying the boundary conditions (3.6) at $r = 1$, we have, after a lengthy but straightforward calculation, an inhomogeneous version of (3.11):

$$\begin{bmatrix} k_i K_1' & -\mathscr{A}_0 \\ (\omega_0+i)K_1 & J_1(\eta_1) \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \beta_1 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}, \tag{3.16a}$$

$$\begin{bmatrix} k_i K_1' & -\bar{\mathscr{A}}_0 \\ (\omega_0-i)K_1 & J_1(\eta_2) \end{bmatrix} \begin{Bmatrix} \bar{\alpha}_1 \\ \bar{\beta}_1 \end{Bmatrix} = \begin{Bmatrix} \bar{F}_1 \\ \bar{F}_2 \end{Bmatrix}, \tag{3.16b}$$

where the inhomogeneous terms F_1, $\bar{F}_1$, F_2 and $\bar{F}_2$ are functions of ω_1, ω_0, k_i and of arbitrary coefficients β_0 and $\bar{\beta}_0$; these functions are written out in appendix B.

Solvability when the determinant of (3.16*a*) and/or (3.16*b*) is zero requires that the forcing terms be orthogonal to the solution of the adjoint problem. This requirement leads to the following conclusion.

If (ω_0, k_i) are such that only $a(\omega_0, k_i) = 0$, then ω_1 can be directly determined from the orthogonality condition $k_i K_1' F_2 - (\omega_0+i) K_1 F_1 = 0$; also $\bar{\beta}_0 = 0$. If (ω_0, k_i) are such that only $b(\omega_0, k_i) = 0$, orthogonality requires

$$k_i K_1' \bar{F}_2 - (\omega_0 - i) K_1 \bar{F}_1 = 0;$$

also $\beta = 0$. For these two cases, ω_1 can be found by inspection to be purely imaginary.

If (ω_0, k_i) are such that $a(\omega_0, k_i) = 0$ and $b(\omega_0, k_i) = 0$ *simultaneously*, then we have

$$k_i K_1' F_2 - (\omega_0+i) K_1 F_1 = 0, \quad k_i K_1' \bar{F}_2 - (\omega_0 - i) K_1 \bar{F}_1 = 0. \tag{3.17}$$

Equation (3.17) can be rewritten using the expressions for the F's of appendix B as

$$[\omega_1 f + k_i \tilde{k} g]\beta_0 + h\bar{\beta}_0 = 0, \quad \bar{h}\beta_0 + [\omega_1 \bar{f} + k_i \tilde{k}\bar{g}]\bar{\beta}_0 = 0, \tag{3.18}$$

where the expressions for f, g, h, $\bar{f}$, $\bar{g}$ and $\bar{h}$ appear in appendix C.

For (3.18) to have a non-trivial solution, its determinant must equal zero; this requires

$$f\bar{f}\omega_1^2 + k_i \tilde{k}(f\bar{g} + \bar{f}g)\omega_1 + k_i^2 \tilde{k}^2 \bar{g}g - h\bar{h} = 0, \tag{3.19}$$

which determines ω_1 as

$$\omega_1 = \{-k_i\tilde{k}(f\bar{g} + g\bar{f}) \pm [k_i^2\tilde{k}^2(f\bar{g}+g\bar{f})^2 - 4f\bar{f}(k_i^2\tilde{k}^2 g\bar{g} - h\bar{h})]^{\frac{1}{2}}\}/2f\bar{f}. \tag{3.20}$$

The growth rate is given by the real part of ω_1. Inspection of (3.20) and the numerical results presented below shows that the vortex is unstable at many but not all of these (ω_0, k_i) points. Thus instability can, but does not necessarily, occur for (ω_0, k_i) combinations for which $a(\omega_0, k_i)$ and $b(\omega_0, k_i)$ are simultaneously zero. Moore & Saffman have also obtained this condition but did not obtain results for a specific case.

The physical explanation for the instability is that this condition,

$$a(\omega_0, k_i) = 0 \quad \text{and} \quad b(\omega_0, k_i) = 0$$

simultaneously, identifies two eigenmodes of the same wavenumber and equal but opposite frequencies, which can produce an oscillatory standing wave on the line vortex. Since this wave maintains a constant angular orientation, it can diverge in a flow with strain. At the intersection points, i.e. $\tilde{k} = 0$, we have the maximum amplification rate

$$\omega_1^2 = h\bar{h}/f\bar{f}.$$

Numerical results for several points are

$$\begin{aligned}
\omega_1 &= 0{\cdot}57 &&\text{at}\quad \omega_0 = 0, && k_i = 2{\cdot}5,\\
\omega_1 &= 0{\cdot}0236 &&\text{at}\quad \omega_0 = \pm 0{\cdot}5, && k_i = 1{\cdot}6,\\
\omega_1 &= 0{\cdot}0041 &&\text{at}\quad \omega_0 = \pm 0{\cdot}405, && k_i = 1{\cdot}27,\\
\omega_1 &= 0{\cdot}021i &&\text{at}\quad \omega_0 = \pm 0{\cdot}191, && k_i = 3{\cdot}24,\\
\omega_1 &= 0{\cdot}023i &&\text{at}\quad \omega_0 = \pm 0{\cdot}135, && k_i = 5{\cdot}25.
\end{aligned}$$

Therefore the instability exists at several such points including $\omega_0 = 0$; this gives some support to the simple condition of instability $\omega = 0$ for some k, postulated by Widnall *et al.* (1974). These points of possible instability are determined by the crossings of the curves of the dispersion relations, shown in figure 1. Note that, from numerical calculations, ω_1 is imaginary at some of these crossing points: the flow is stable to these disturbances. In the simple analysis of Widnall *et al.* (1974) the amplification rate was unity; the present more complete analysis gives 1·14 under the same condition. The previous model was, of course, only a heuristic explanation and not a full three-dimensional analysis.

The flow is unstable in a small band of wavenumbers $|k-k_i| = \epsilon\tilde{k}$ around an unstable crossing point (ω_0, k_i). In general, for $\omega_0 \neq 0$, it can be shown that ω_1 is positive real as long as

$$\tilde{k}^2 < \frac{4f\bar{f}h\bar{h}}{k_i^2[4\bar{f}f\bar{g}g-(f\bar{g}+\bar{f}g)^2]}. \tag{3.21}$$

For the case $\omega_0 = 0$, we have $f\bar{g}+\bar{f}g = 0$, and $\bar{f} = -f$ (purely imaginary), $\bar{g} = g$ (real) and $h = \bar{h}$ (real); then

$$\omega_1^2 = h^2/f^2$$

is a real number indicating the rate of growth.

In the limiting case $\omega_0 = 0$, $k_i \to 0$, then $k_i g = 0$, $h = \frac{1}{2}\sqrt{3}$ and $f = \sqrt{3}$, so that we obtain $\omega_1^2 = \frac{1}{4}$, in agreement with the results of Moore & Saffman (1971).

For $\omega_0 = 0$, $\omega_1^2 > 0$ whenever

$$|\tilde{k}| < \hat{k} = h/k_i g.$$

Some numerical values for the width $\hat{k}$ of the region of instability are

$$\begin{aligned}
\omega_{1\max} &= 0{\cdot}5708, && \hat{k} < 2{\cdot}14 &&\text{at}\quad \omega_0 = 0, && k_i = 2{\cdot}5,\\
\omega_{1\max} &= 0{\cdot}5695, && \hat{k} < 3{\cdot}5 &&\text{at}\quad \omega_0 = 0, && k_i = 4{\cdot}35,\\
\omega_{1\max} &= 0{\cdot}004, && \hat{k} < 0{\cdot}012 &&\text{at}\quad \omega_0 = \pm 0{\cdot}4, && k_i = 1{\cdot}27,\\
\omega_{1\max} &= 0{\cdot}024, && \hat{k} < 0{\cdot}092 &&\text{at}\quad \omega_0 = \pm 0{\cdot}5, && k_i = 1{\cdot}6.
\end{aligned}$$

In dimensional form, the amplification rate α of the instability for the case $\omega_0 = 0$ is

$$\alpha = \frac{2e}{\Omega}\frac{1}{f}(h^2 - k_i^2\tilde{k}^2g^2)^{\frac{1}{2}}. \tag{3.22}$$

The width $\hat{k}$ of the zone of instability and the maximum rate of amplification are proportional to the ratio of the rate of strain e to the vorticity. The numerical results obtained indicate that both the amplification rate and the width of the unstable region are much larger at $\omega_0 = 0$ than at the crossing points with $\omega_0 \neq 0$.

The extension of the analysis to the vortex ring remains to be done.

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Appendix A. Definition of functions introduced in (3.15)

The functions H_i and $\bar{H}_i$ are as follows:

$$H_1 \equiv \frac{-4k_i^2}{(\omega_0+i)^3}\left[\frac{J_1(\eta_1 r)}{\eta_1^2} - \frac{rJ_0(\eta_1 r)}{\eta_1}\right],$$

$$H_2 \equiv \frac{(\omega_0-i)(\omega_0+3i)}{(\omega_0+i)^2}\left[\frac{J_1(\eta_1 r)}{\eta_1^2} - \frac{rJ_0(\eta_1 r)}{\eta_1}\right],$$

$$H_3 \equiv i\left\{\frac{2\pi\omega_0 k_i^2\eta_1}{(\omega_0^2+1)^2}[\bar{S}_1(r)Y_1(\eta_2 r) - \bar{S}_2(r)J_1(\eta_2 r)] + \frac{k_i^2(\omega_0+3i)}{(\omega_0+i)^2}\frac{J_1(\eta_1 r)}{\eta_1^2-\eta_2^2}\right\},$$

$$\bar{H}_1 \equiv \frac{-4k_i^2}{(\omega_0-i)^3}\left[\frac{J_1(\eta_2 r)}{\eta_2^2} - \frac{rJ_0(\eta_2 r)}{\eta_2}\right],$$

$$\bar{H}_2 \equiv \frac{(\omega_0+i)(\omega_0-3i)}{(\omega_0-i)^2}\left[\frac{J_1(\eta_2 r)}{\eta_2^2} - \frac{rJ_0(\eta_2 r)}{\eta_2}\right],$$

$$\bar{H}_3 \equiv i\left\{\frac{-2\pi\omega_0 k_i^2\eta_2}{(\omega_0^2+1)^2}[S_1(r)Y_1(\eta_1 r) - S_2(r)J_1(\eta_1 r)] + \frac{k_i^2(\omega_0-3i)}{(\omega_0-i)^2}\frac{J_1(\eta_2 r)}{\eta_1^2-\eta_2^2}\right\},$$

where J_0 and J_1 are Bessel functions, Y_1 is a modified Bessel function and

$$S_1(r) \equiv \int^r t^2 J_0(\eta_2 t)J_1(\eta_1 t)\,dt, \quad \bar{S}_1(r) \equiv \int^r t^2 J_0(\eta_1 t)J_1(\eta_2 t)\,dt,$$

$$S_2(r) \equiv \int^r t^2 J_0(\eta_2 t)Y_1(\eta_1 t)\,dt, \quad \bar{S}_2(r) \equiv \int^r t^2 J_0(\eta_1 t)Y_1(\eta_2 t)\,dt.$$

Appendix B. Definitions of functions introduced in (3.16)

The functions F_i and $\bar{F}_i$ evaluated at $r = 1$ are as follows:

$$F_1 \equiv [-k_i\tilde{k}(1+k_i^{-2})K_1(k_i)]\alpha_0 + [\tfrac{1}{2}K_1(k_i) - \tfrac{1}{4}k_i^2K_1''(k_i)]\bar{\alpha}_0 + [\omega_1 A_1 + k_i\tilde{k}A_2]\beta_0 + [\bar{A}_3 - \tfrac{1}{2}i\bar{\mathscr{B}}_0 + \tfrac{1}{4}\bar{\mathscr{A}}_0']\bar{\beta}_0,$$

$$F_2 \equiv \left[-\omega_1 K_1(k_i) + (\omega_0+i)k_i\tilde{k}\left(\frac{K_0(k_i)}{k_i} + \frac{K_1(k_i)}{k_i^2}\right)\right]\alpha_0 + [-\tfrac{1}{2}iK_1(k_i) - \tfrac{1}{4}(\omega_0+i)k_iK_1'(k_i)]\bar{\alpha}_0 + [-\omega_1 H_1 - k_i\tilde{k}H_2]\beta_0 + [-\bar{H}_3 - \tfrac{1}{4}\eta_2 J_1'(\eta_2)]\bar{\beta}_0,$$

$$\bar{F}_1 \equiv [-k_i\tilde{k}(1+k_i^{-2})K_1(k_i)]\bar{\alpha}_0 + [\tfrac{1}{2}K_1(k_i) - \tfrac{1}{4}k_i^2K_1''(k_i)]\alpha_0 + [\omega_1\bar{A}_1 + k_i\tilde{k}\bar{A}_2]\bar{\beta}_0 + [A_3 + \tfrac{1}{2}i\mathscr{B}_0 + \tfrac{1}{4}\mathscr{A}_0']\beta_0,$$

$$\bar{F}_2 \equiv \left[-\omega_1 K_1(k_i) + (\omega_0-i)k_i\tilde{k}\left(\frac{K_0(k_i)}{k_i} + \frac{K_1(k_i)}{k_i^2}\right)\right]\bar{\alpha}_0 + [\tfrac{1}{2}K_1(k_i) - \tfrac{1}{4}(\omega_0-i)k_iK_1'(k_i)]\alpha_0 + [-\omega_1\bar{H}_1 - k_i\tilde{k}\bar{H}_2]\bar{\beta}_0 + [-H_3 - \tfrac{1}{4}\eta_1 J_1'(\eta_1)]\beta_0,$$

where

$A_1 \equiv [-(\omega_0+i)H_1'-2ir^{-1}H_1-(\omega_0+i)\mathscr{A}_0-2\mathscr{B}_0]/[4+(\omega_0+i)^2],$

$A_2 \equiv [-(\omega_0+i)H_2'-2ir^{-1}H_2]/[4+(\omega_0+i)^2],$

$A_3 \equiv [-(\omega_0-i)H_3'+2ir^{-1}H_3+i(\omega_0-i)(\frac{1}{2}r\mathscr{A}_0'+\mathscr{A}_0)+ir\mathscr{B}_0'+2\mathscr{A}_0]/[4+(\omega_0-i)^2],$

$\bar{A}_1 \equiv [-(\omega_0-i)\bar{H}_1'+2ir^{-1}\bar{H}_1-(\omega_0-i)\bar{\mathscr{A}}_0-2\bar{\mathscr{B}}_0]/[4+(\omega_0-i)^2],$

$\bar{A}_2 \equiv [-(\omega_0-i)\bar{H}_2'+2ir^{-1}\bar{H}_2]/[4+(\omega_0-i)^2],$

$\bar{A}_3 \equiv [-(\omega_0+i)\bar{H}_3'-2ir^{-1}\bar{H}_3-i(\omega_0+i)(\frac{1}{2}r\bar{\mathscr{A}}_0'+\bar{\mathscr{A}}_0)-ir\bar{\mathscr{B}}_0'+2\bar{\mathscr{A}}_0]/[4+(\omega_0+i)^2],$

$\mathscr{A}_0 \equiv [-(\omega_0+i)\eta_1 J_0(\eta_1 r)+(\omega_0-i)J_1(\eta_1 r)/r]/[4+(\omega_0+i)^2],$

$\mathscr{B}_0 \equiv [2\eta_1 J_0(\eta_1 r)-i(\omega_0-i)J_1(\eta_1 r)/r]/[4+(\omega_0+i)^2],$

$\bar{\mathscr{A}}_0 \equiv [-(\omega_0-i)\eta_2 J_0(\eta_2 r)+(\omega_0+i)J_1(\eta_2 r)/r]/[4+(\omega_0-i)^2],$

$\bar{\mathscr{B}}_0 \equiv [2\eta_2 J_0(\eta_2 r)+i(\omega_0+i)J_1(\eta_2 r)/r]/[4+(\omega_0-i)^2].$

Here J_0 and J_1 are Bessel functions, K_0 and K_1 are modified Bessel functions and from (3.12)

$$\alpha_0 = \frac{\mathscr{A}_0}{kK_1'}\beta_0, \quad \bar{\alpha}_0 = \frac{\bar{\mathscr{A}}_0}{kK_1'}\bar{\beta}_0.$$

Appendix C. Definitions of functions introduced in (3.18)

The functions in (3.18) evaluated at $r = 1$ are as follows:

$$f \equiv -\mathscr{A}_0 K_1(k_i) - k_i K_1'(k_i) H_1 - (\omega_0+i) K_1(k_i) A_1,$$

$$g \equiv (\omega_0+i)\mathscr{A}_0\left[\frac{K_0(k_i)}{k_i}+\frac{K_1(k_i)}{k_i^2}+\frac{(1+k_i^2)K_1^2(k_i)}{k_i^3K_1'(k_i)}\right] - k_iK_1'(k_i)H_2-(\omega_0+i)K_1(k_i)A_2,$$

$$h \equiv \left[-\frac{(\omega_0+i)}{4}\left(k_iK_1'(k_i)+\frac{2K_1^2(k_i)-K_1(k_i)k_i^2K_1''(k_i)}{k_iK_1'(k_i)}\right)-\tfrac{1}{2}iK_1(k_i)\right]\bar{\mathscr{A}}_0 - k_iK_1'(k_i)[\bar{H}_3+\tfrac{1}{4}\eta_2J_1'(\eta_2)]-(\omega_0+i)K_1(k_i)(\bar{A}_3-\tfrac{1}{2}i\bar{\mathscr{B}}_0+\tfrac{1}{4}\bar{\mathscr{A}}_0'),$$

$$\bar{f} \equiv -\bar{\mathscr{A}}_0 K_1(k_i) - k_i K_1'(k_i) \bar{H}_1 - (\omega_0-i) K_1(k_i) \bar{A}_1,$$

$$\bar{g} \equiv (\omega_0-i)\bar{\mathscr{A}}_0\left[\frac{K_0(k_i)}{k_i}+\frac{K_1(k_i)}{k_i^2}+\frac{(1+k_i^2)K_1^2(k_i)}{k_i^3K_1'(k_i)}\right] - k_iK_1'(k_i)\bar{H}_2-(\omega_0-i)K_1(k_i)\bar{A}_2,$$

$$\bar{h} \equiv \left[-\frac{(\omega_0-i)}{4}\left(k_iK_1'(k_i)+\frac{2K_1^2(k_i)-K_1(k_i)k_i^2K_1''(k_i)}{k_iK_1'(k_i)}\right)+\tfrac{1}{2}iK_1(k_i)\right]\mathscr{A}_0 - k_iK_1'(k_i)[H_3+\tfrac{1}{4}\eta_1J_1'(\eta_1)]-(\omega_0-i)K_1(k_i)(A_3+\tfrac{1}{2}i\mathscr{B}_0+\tfrac{1}{4}\mathscr{A}_0').$$

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